

MINIMAL HYPERSURFACES IN THE BALL WITH FREE BOUNDARY

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ABSTRACT. In this note we use the strong maximum principle and integral estimates prove two results on minimal hypersurfaces $F : M^n \rightarrow \mathbb{R}^{n+1}$ with free boundary on the standard unit sphere. First we show that if F is graphical with respect to any Killing field, then $F(M^n)$ is a flat disk. Second, if $M^n = \mathbb{D}^n$ is a disk, we show that either $F(M^n)$ is a flat disk, or the supremum of the curvature squared on the interior is bounded below by n times the infimum of the curvature squared on the boundary.

These may be combined to give an impression of the curvature of non-flat minimal disks of higher intrinsic dimension.

1. INTRODUCTION

Recently, minimal surfaces with free boundary have received much attention. A landmark result due to Nitsche is:

Theorem (Theorem 1 in [17]). *Let $F : \mathbb{D}^2 \rightarrow \mathbb{R}^3$ be a proper branched minimal immersion with free boundary on the standard unit sphere. Then $F(\mathbb{D}^2)$ is a flat disk.*

The proof exploits the Hopf differential via complex analysis.

There has been much work extending this result in various directions. This activity has yielded some excellent results, as a small selection we refer to [3, 5, 7, 8, 11, 14, 15, 23]. Fraser-Schoen [9] made a recent influential contribution, that includes an extension of Nitsche's Theorem to arbitrary codimension.

In this note we study the higher dimensional analogue of this problem, for minimal hypersurfaces with free boundary in the unit ball. Although there is a wealth of knowledge available on the problem for $n = 2$, in the higher dimensional case results are much more scarce. One reason for this is that incredibly powerful complex analytical techniques that apply for surfaces do not seem to carry over to hypersurfaces. Nevertheless, progress continues to be made: see Sargent [19] for some new index bounds for minimal hypersurfaces with free boundary, Mondino-Spadaro [16] for a new characterisation of free boundary minimal submanifolds, and Li plus collaborators [13, 12] for far-reaching min-max and regularity theory, including an extension of the classical program of Almgren [1, 2] (see Pitts [18] and Schoen-Simon [20] for further classical theory) to the case of minimal hypersurfaces with free boundary, for example.

Our contribution is on the question of uniqueness of minimal embedded n -disks. We prove:

Theorem 1.1 (Uniqueness of graphical disks). *Let $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth graphical minimal hypersurface with free boundary on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then $M^n = \mathbb{D}^n$ and $F(M^n)$ is a standard flat disk.*

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In fact, we prove something stronger than this. Let $\nu^M : M^n \rightarrow \mathbb{R}^{n+1}$ be a unit normal vector field along F . Let $s_V : M^n \rightarrow \mathbb{R}$ be the function given by

$$(1) \quad s_V(x) := \langle \nu^M(x), V(F(x)) \rangle$$

where $V : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is a Killing vector field. We prove in Section 4 that: If $s_V > 0$, then $F(M^n)$ is a flat disk. Applying this result with V a translation yields Theorem 1.1 as a corollary.

The main result of Ambrozio-Nunes [3] is that if $M^n = \mathbb{D}^n$ and $|A|^2 \langle F, \nu^M \rangle^2 \leq 2$, then $F(\mathbb{D}^n)$ is flat. If we allow more freedom in the domain of F , the only other possibility is that at some point $|A|^2 \langle F, \nu^M \rangle^2 = 2$ and $F(\mathbb{S} \times (a, b))$ is a critical catenoid. This result is special to the case of surfaces, but does indicate a kind of ‘curvature gap’ phenomenon at work. Our second result moves also in this direction.

Theorem 1.2 (Curvature gap). *Let $F : \mathbb{D}^n \rightarrow \mathbb{R}^{n+1}$ be a smooth minimal immersed n -disk with free boundary on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Suppose that $F(\mathbb{D}^n)$ is not a standard flat disk. Then*

$$(i) \quad \inf_{\mathbb{D}^n} |A|^2 > 0, \quad \text{and}$$

$$(ii) \quad \left(\sup_{\mathbb{D}^n} |A|^2 \right)^2 > n \inf_{\partial \mathbb{D}^n} |A|^2.$$

Remark. One has automatically that $\sup_{\mathbb{D}^n} |A|^2 \geq \inf_{\partial \mathbb{D}^n} |A|^2$, and so our estimate (ii) above gives new information only when $\inf_{\partial \mathbb{D}^n} |A|^2 \in (0, n]$.

Alternative (i) was at least known to Simons [21]. It also holds in greater generality than given above:

Proposition 1.3. *Let $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth minimal immersed hypersurface, with or without boundary. Then one of the following holds:*

- $F(M^n)$ is contained in an n -dimensional affine subspace of \mathbb{R}^{n+1} ; or
- $\inf_{M^n} |A|^2 > 0$.

If $n = 2$, we may improve alternative (ii) in Theorem 1.2:

Theorem 1.4 (Sharper bounds for surfaces). *For $n = 2$, under the hypotheses of Theorem 1.2 we have*

$$\left(\sup_{\mathbb{D}^2} |A|^2 \right)^2 > 2n \inf_{\partial \mathbb{D}^2} |A|^2.$$

Finally, we wish to note that minimal hypersurfaces with free boundary on \mathbb{S}^n achieve equality in the isoperimetric inequality. Similar results have been obtained in [10]. Our proof is well known, see for example Brendle [4], where the same proof is used for surfaces.

Proposition 1.5 (Isoperimetric equality). *Let $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth minimal immersed hypersurface with free boundary on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then*

$$n|M| = |\partial M|,$$

where $|M|$ and $|\partial M|$ are the volume of the hypersurface and the volume of its boundary respectively.

2. SETTING

Consider the standard unit sphere in Euclidean space $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$. We use $\nu^{\mathbb{S}^n} : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ to denote its outer normal vectorfield. Let M^n be a smooth, orientable n -dimensional Hausdorff paracompact manifold with boundary ∂M^n . Let g be a Riemannian metric on M^n . Set $M := F(M^n) \subset \mathbb{R}^{n+1}$ where $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth isometric immersion satisfying

$$(2) \quad \begin{aligned} \partial_N M &\equiv F(\partial_N M^n) = M \cap \mathbb{S}^n, \\ \langle \nu^M, \nu^{\mathbb{S}^n} \circ F \rangle(p) &= 0, \quad \forall p \in \partial_N M^n. \end{aligned}$$

Since F is isometric, the Riemannian structure induced by the embedding F is the same as that given by g , that is $(M^n, g) = (M^n, F^* \delta)$, where δ is the standard metric on \mathbb{R}^n .

Let us denote by $A^M : TM \times TM \rightarrow \mathbb{R}$ the second fundamental form of M with components given by h_{ij} where $1 \leq i, j \leq n$ where $h_{ij} = A^M(\tau_i, \tau_j)$ for two sections τ_i and τ_j in TM . For \mathbb{S}^n we have $A^{\mathbb{S}^n} : T\mathbb{S}^n \times T\mathbb{S}^n \rightarrow \mathbb{R}$ the second fundamental form with components $h_{ij}^{\mathbb{S}^n}$ for $1 \leq i, j \leq n$.

3. AUXILLIARY EQUATIONS

Let us define the quantities we use in the proof of the uniqueness theorem. Recall the function $s_V : M \rightarrow \mathbb{R}$ defined in (1) above. When V is a translation, following [6] we term s_V the *graph quantity*. If, up to reparametrisation,

$$F(x) = (x, u(x)) = x_i e_i + u(x) V,$$

then one can relate the gradient of the associated scalar function u to the reciprocal of the graph quantity s_V . This implies that a lower bound on s is equivalent to a gradient bound for u . Throughout this section we assume that $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth minimal immersed hypersurface.

Lemma 3.1. *The quantity $s_V : M^n \rightarrow \mathbb{R}$ satisfies*

$$\Delta^M s_V = -|A^M|^2 s_V.$$

The squared reciprocal of s_V denoted by $v_V^2 = \frac{1}{s_V^2}$ also satisfies an elliptic equation (see [6]):

Lemma 3.2. *The quantity $v_V^2 : M^n \rightarrow \mathbb{R}$ satisfies*

$$\Delta^M v_V^2 = 2|A^M|^2 v_V^2 + 6|\nabla v_V|^2.$$

The support function is defined as $u : M \rightarrow \mathbb{R}$:

$$u(x) := \langle F(x), \nu_M(x) \rangle.$$

It is strictly positive for convex bodies (that contain the origin) and vanishes on linear subspaces of \mathbb{R}^{n+1} . On a minimal hypersurface, it satisfies the following equation:

Lemma 3.3. *The quantity $u^2 : M \rightarrow \mathbb{R}$ satisfies*

$$\Delta^M u^2 = -2|A|^2 u^2 + 2|\nabla u|^2.$$

We also require the following evolution equation for the product of v_V^2 and u^2 .

Lemma 3.4. *The quantity $\mathcal{Q} = u^2 v_V^2 : M \rightarrow \mathbb{R}$ satisfies*

$$\Delta^M Q \geq 2 \frac{\nabla v_V}{v_V} \nabla Q.$$

Proof. We compute using Lemmata 3.2 and 3.3

$$\begin{aligned} \Delta^M Q &= v_V^2 \Delta^M u^2 + u^2 \Delta^M v_V^2 + 2 \nabla v_V^2 \nabla u^2 \\ &= 2 v_V^2 |\nabla u|^2 + 6 u^2 |\nabla v_V|^2 + 2 \nabla v_V^2 \nabla u^2. \end{aligned}$$

Separately we can transform the mixed gradient term into a gradient of the Q quantity and extra terms as follows:

$$2 \nabla v_V^2 \nabla u^2 = \nabla v_V^2 \nabla u^2 + 4 u v_V \nabla v_V \nabla u = 2 \frac{\nabla v_V}{v_V} \nabla Q - 4 |\nabla v_V|^2 u^2 + 4 u v_V \nabla v_V \nabla u.$$

Replacing into the above completes the proof:

$$\begin{aligned} \Delta^M Q &= 2 \frac{\nabla v_V}{v_V} \nabla Q + 2 v_V^2 |\nabla u|^2 + 6 u^2 |\nabla v_V|^2 - 4 |\nabla v_V|^2 u^2 + 4 u v_V \nabla v_V \nabla u \\ &= 2 \frac{\nabla v_V}{v_V} \nabla Q + 2 v_V^2 |\nabla u|^2 + 2 u^2 |\nabla v_V|^2 + 4 u v_V \nabla v_V \nabla u \\ &\geq 2 \frac{\nabla v_V}{v_V} \nabla Q. \end{aligned}$$

□

The following calculation is standard.

Lemma 3.5. *The square of the second fundamental form satisfies the equation*

$$\frac{1}{2} \Delta^M |A|^2 = |\nabla A|^2 - |A|^4.$$

We also require the boundary derivative of the second fundamental form. For any boundary point $X \in \partial M$ the Neumann boundary condition allows us to chose a basis $\{\tau_1, \dots, \tau_n\}$ of the tangent space $T_X M$ such that $\tau_i \in T \partial M \cap T \mathbb{S}^n$ for all $i \in \{1, \dots, n-1\}$ and $\tau_n = \nu^{\mathbb{S}^n}$ at X .

Let $A^M(\tau_i, \tau_j) = (h_{ij})_{1 \leq i, j \leq n}$ and $A^{\mathbb{S}^n}(\tau_i, \tau_j) = (h_{ij}^{\mathbb{S}^n})_{1 \leq i, j \leq n}$ be the second fundamental forms of M and \mathbb{S}^n respectively. Note that for any choice of orthonormal basis of the tangent space of \mathbb{S}^n the second fundamental form of \mathbb{S}^n is diagonal. It was shown by Stahl [22] that on the boundary we have

$$h_{in} = h_{in}^{\mathbb{S}^n} = 0 \quad \text{for all } i = 1, \dots, n-1.$$

Since $n-1$ of the tangent vectors are on the closed submanifold $\partial M \subset \mathbb{S}^n$ we can choose our basis above such that on the boundary

$$h_{ij} = 0 \quad \text{if } i \neq j, \quad \text{and} \quad h_{ij}^{\mathbb{S}^n} = \delta_{ij} \quad \text{if } i \neq j.$$

We also need the following boundary relations, again due to Stahl [22, Theorem 3.7].

For the remainder of this section we additionally assume that F has free boundary on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

Lemma 3.6 (Normal derivatives). *Along the boundary in an orthonormal basis as described above the following hold:*

$$\begin{aligned}\nabla_n H &= H, \\ \nabla_n h_{ii} &= (h_{nn} - h_{ii}) \quad \text{for all } i = 1, \dots, n-1.\end{aligned}$$

We are now ready to state our result about the boundary derivative of the second fundamental form squared.

Proposition 3.7 (Normal derivative of $|A|^2$). *Along the boundary in an orthonormal basis as described above the following hold:*

$$\nabla_n |A|^2 = -2|A|^2 - 2nh_{nn}^2.$$

Proof. The result follows by using Lemma 3.6 and

$$\nabla_n h_{nn} = \nabla_n H - \sum_{i=1}^{n-1} \nabla_n h_{ii}.$$

□

4. MINIMAL GRAPHICAL HYPERSURFACES WITH FREE BOUNDARY ARE FLAT DISKS

Now assume that

$$(3) \quad \inf_M s_V \geq C_0,$$

for some constant $C_0 > 0$. We shall now prove:

Theorem 4.1. *Let $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth minimal hypersurface with free boundary on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Assume (3) for some Killing field V . Then $F(M^n)$ is a standard flat disk.*

Proof. Apply the elliptic maximum principle to the evolution of $Q \geq 0$ to see that there is no interior maxima. On the boundary we note that $Q = u = 0$ using $F = \nu^{\mathbb{S}^n}$ and the Neumann boundary condition.

Therefore $Q = u^2 v_V^2 = 0$ everywhere on M . By hypothesis $v_V \neq 0$ and so the support function vanishes on M . Using now the smoothness assumption, non-flat cones are ruled out, leaving only the possibility that M is a flat disk. □

Proof of Theorem 1.1. By hypothesis there exists a translation vector field V such that $s_V \geq C_0$, implying condition (3). The theorem follows by applying Theorem 4.1. □

Remark. The proof also works if we replace \mathbb{S}^n by a sphere of any fixed radius R . The bounded curvature and graphicality condition are required to bound the coefficient of the gradient term in the evolution of Q .

Remark. It is only necessary to assume that $s_V > 0$ on the interior, it may a-priori assume zeroes on the boundary.

5. CURVATURE GAP

Let us first prove Proposition 1.3.

Proof of Proposition 1.3. Lemma 3.5 gives

$$\frac{1}{2}\Delta^{M_t}|A|^2 = |\nabla A|^2 - |A|^4.$$

Suppose there is a point x in the interior of M such that $|A|^2(x) = 0$ and $|A|^2 > 0$ in a neighbourhood around $x \in \mathcal{U} \subset M$. This makes x a point where a minimum for $|A|^2$ is attained in the interior. We can compute $|\nabla A|^2 = |\nabla|A||^2 = 0$ since $|A|$ also attains a strict minimum at x , which implies that at x we have $\Delta^M|A|^2 = 0$ which contradicts the strong minimum principle, thus the conclusion. \square

We also require the isoperimetric equality stated in our introduction so we derive it here.

Proof of Proposition 1.5. This is a simple routine computation using the divergence theorem. We include it here for completeness.

$$0 = \int_M \langle -H\nu^M, F \rangle d\mu = \int_M \langle \Delta F, F \rangle d\mu = - \int_M |\nabla F|^2 + \int_{\partial M} \langle \nabla_{\nu_{S^n}} F, F \rangle dS,$$

where we have used minimality in the first equality, the evolution of the position vector in the second and the divergence theorem in the last. We denote the unit outer pointing normal to S^n by ν^{S^n} . Due to the perpendicular boundary condition of the minimal hypersurface we also have $\nu^{S^n} = \nu^{\partial M}$. Note that $\nu^{S^n} = F$ on $\partial M \subset S^n$. Also note that $|\nabla F|^2 = n$ and that $\nabla_{\nu_{S^n}} F = F$ giving us that $\langle \nabla_{\nu_{S^n}} F, F \rangle = 1$ on $\partial M \subset S^n$. This completes our proof. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The first part of the theorem follows directly from Proposition 1.3. For the second we will use divergence theorem, this time on the second fundamental form squared. Assume for the moment that

$$(4) \quad \left(\sup_{\mathbb{D}^n} |A|^2 \right)^2 \leq n \inf_{\partial \mathbb{D}^n} |A|^2.$$

Now compute

$$\int_M \Delta |A|^2 d\mu = \int_{\partial M} \nabla_{\nu_{S^n}} |A|^2 dS = - \int_{\partial M} 2|A|^2 dS - \int_{\partial M} 2nh_{nn}^2 dS \leq - \int_{\partial M} 2|A|^2 dS,$$

where we have used Proposition 3.7 in the last equality. For $n = 2$ we are able to use the term $\int_{\partial M} 2nh_{nn}^2 dS$; we do this below in the proof of Theorem 1.4. Unfortunately we are unable to fully absorb the last term before the inequality into $|A|^2$ as can be done for $n = 2$ (see below for proof of Corollary 1.4) since nothing is preventing h_{nn} from vanishing even though the full $|A|^2$ is not.

Lemma 3.5 implies

$$\int_M |\nabla A|^2 d\mu = \int_M |A|^4 d\mu - \int_{\partial M} 2|A|^2 dS \leq |M| \left(\sup_{\mathbb{D}^n} |A|^2 \right)^2 - |\partial M| \inf_{\partial \mathbb{D}^n} |A|^2.$$

Now (4) combined with Proposition 1.5 yields

$$\int_M |\nabla A|^2 d\mu \leq 0.$$

This gives us that $|\nabla A|^2 \equiv 0$ everywhere on M , implying that all principal curvatures of F are constant, and so that M is a part of a plane or a sphere. But M is minimal thus part of a plane. The only plane with perpendicular boundary condition are equatorial disks. But this is a contradiction with our hypothesis that $F(\mathbb{D}^n)$ is not a standard flat disk.

Therefore the assumption (4) is false, and we conclude part (ii) of the theorem. \square

We finish by showing that the bound is sharper for the case of surfaces.

Proof of Theorem 1.4. In the case of surfaces minimality implies that $h_{11} = -h_{22}$ thus we can use the second negative term from the boundary making $-2|A|^2 - 2nh_{nn}^2 = -4|A|^2$ when $n = 2$. Carrying this improvement through yields the theorem. \square

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